**Lesson 0**

Introduction to Complex Numbers

The set of all complex numbers is essentially a two-dimensional extension of the field of real numbers. By a complex number we mean a number comprising a real and an imaginary part.

It can be written in the form ;

where and are real numbers, and i is postulated to be the imaginary unit with the property

Complex numbers help in two areas:

- application in real-world applications

- simplifying mathematics

**Cartesian Form**

or

Where and are both real numbers and is known as the imaginary unit and satisfies .

The number is called “the real part of ”

The number is called “the imaginary part of ”

**Integer powers of**

Every integer power of is a member of the set

**Lesson 2**

Graphical Representation

<https://jutanium.github.io/ComplexNumberGrapher/>

Complex numbers are represented on the complex plane (Argand diagram) with:

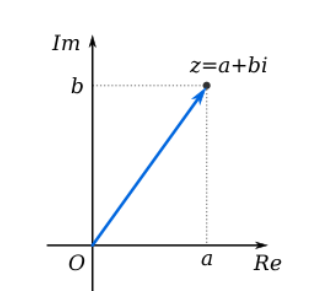
- a “real” (horizontal) axis

- an “imaginary” (vertical) axis

These are written in the form

The number represents a rotation from the real number line

Therefore, a complex number can represent a point, with the real part representing the position on the horizontal, real number line and the imaginary part representing the position on the imaginary or vertical axis.



Operations with complex numbers use the properties of to transform these points.

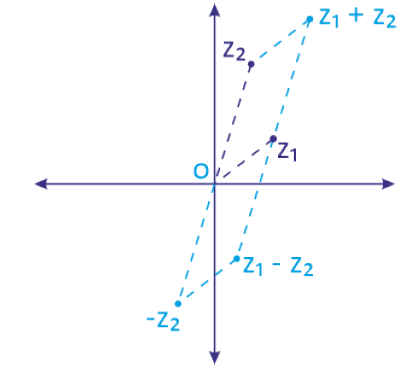
Example: Square the complex number

**Lesson 2**

Operations on complex numbers

**Addition/Subtraction**

*Just add/subtract the corresponding components*

****

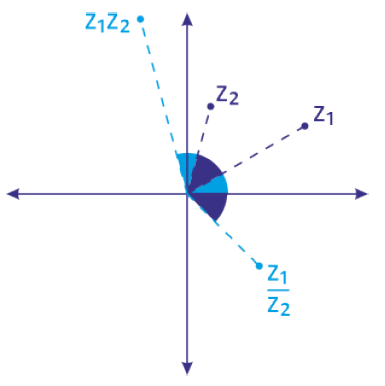
If and

Addition of two complex numbers can be represented as a parallelogram.



**Scalar Multiplication**

If

Scalar multiplication of complex numbers is represented through the stretching principle.

**Multiplication of two complex numbers**

If and

Multiplication of two complex numbers is done through rotation and stretching. The length of is the product of and . The angle of is the sum of and

*Lengths calculated using Pythagoras theorem.*

*Angles calculated using trig.*

**Division by a complex number**

*Multiply by a conjugate (turns denominator into real number)*

If and

*but remember*

**Quadratic formula**

**Lesson 2**

Polar Coordinates

<https://www.cimt.org.uk/projects/mepres/alevel/fpure_ch3.pdf>

Can be expressed as (3,4) on an argand diagram

modulus

Can be expressed as length & direction,

The above is written as:

*modulus of z*

The angle of is

argument

The above is written as

is the argument or phase of

Polar form is just a combination of the above

Example: Write in polar form

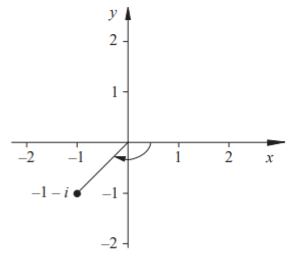
**Modulus**

[1] rewrite in the form

*already in the correct form*

[2] express as

*OA: a line from origin to point a*



**Argument**

[1] find the angle of in OA

*if a and b are positive*

Convert degrees to radians:

Principal value =

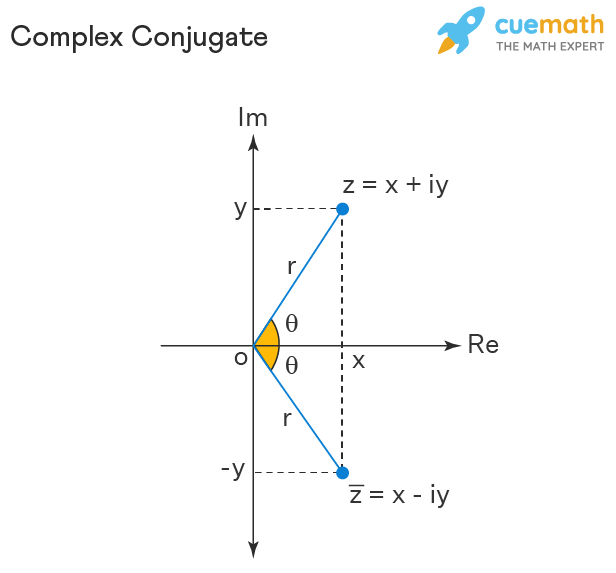
Therefore

**Complex Conjugation**

<https://www.cuemath.com/numbers/complex-conjugate/>

Every complex number has another complex number associated with it, known as the complex conjugate. A complex conjugate of a complex number is another complex number that has the same real part as the original complex number and the imaginary part has the same magnitude but opposite sign. The product of a complex number and its complex conjugate is a real number.

*When we multiply a complex number by its conjugate we get a real number, in other words the imaginary part cancels out*



**Conjugate forms**

**Circular vs Hyperbolic trig**

<https://www.euclideanspace.com/maths/geometry/trig/index.htm>

**Conversions between Circular & Hyperbolic functions**

sin(z) = -i sinh(iz)

cos(z) = -cosh(iz)

tan(z) = -i tanh(iz)

**Hyperbolic identities**

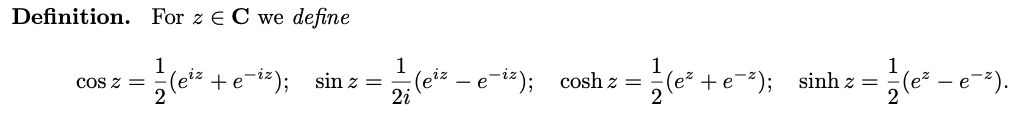
Table

Description automatically generated

Table

Description automatically generated

Circular Identities



Example [NOV 2018]

Let .

If show that

[1] Using complex conjugation

*Let z be the part of the equation that contains any z’s*

*Our goal was to get a term that contained*

*remember*

*Try re-write in terms of*

[1] Substitute known terms

*since*

*Remember that*

Example [NOV 2015]

Let

If , show that

[1] Using complex conjugation

*We already have a semi conjugate*

*remember*

*remember that constants don’t get*

*Try re-write in terms of*

*remember*

*Separate real and imaginary parts*

[2] Simplify using real parts

If then

**Roots of complex numbers**

Suppose that and , where and are real.

Then .

Comparing the real and imaginary parts we see that

and .

Example: Find all those that satisfy

<https://www.maths.ox.ac.uk/system/files/attachments/complex%20%281%29.pdf>

Suppose that and , where and are real.

Then .

Comparing the real and imaginary parts we see that

and .

Example: [NOV 2015]

Find all those that satisfy

[1] Convert / imaginary parts

*parabolic identity:*

­­

*Put into square form for quadratic formula:*

Substitute

Quadratic formula: ;

[2] write in terms of , , and

*Remember that*

circular equation, the period of and is

Let .

Then

Example: [NOV 2015]

Find all those that satisfy

Alternate answer using Complex trig identities

[1] Convert / imaginary parts

*Remember that*

*Also, remember that*

So

and

[2] Real part

circular equation, the period of and is

[3] Imaginary part (use parabolic or circular identities)

*circular identity:*

*Put into square form for quadratic formula:*

Quadratic formula: ;

[4] Let .

Then

Example: [ASS1 2021]

Find all those that satisfy

Alternate answer using Complex trig identities

[1] Convert / imaginary parts

[2] Imaginary part

Circular identity:

*Put into square form for quadratic formula:*

*Quadratic formula:*

­­

[3] Real part

circular equation, the period of and is

[4] Let

**Lesson 3**

Sketching regions in the complex plane

<https://www.youtube.com/watch?v=8gtnZ5xSLuE>

**General case**

and

This represents a “wedge”

(could actually be a disk if and are large enough)

*is a complex number*

*is a positive real number*

*and are real numbers between and*

We want to include all the complex numbers that surround

We also want to rotate between and

*Assume is negative and is positive*

Region of interest: the region we will sketch is somewhere between and



<https://www.maths.usyd.edu.au/MATH1921/r/2018s1/pdf/tut02s.pdf>

<https://www.slader.com/discussion/question/sketch-the-following-sets-and-determine-which-are-domains-a/>

|  |  |  |  |
| --- | --- | --- | --- |
| **Form** | **Example** | **Example Set** | **Type of sketch** |
| and |  | and | “wedge” between and |
|  |  |  | Portion of the entire plane |
|  |  |  | Circle region.  Centered at of radius |
|  |  |  | Circle region. Centered at of radius |
|  |  |  | Region outside circle. Centered at of radius 2 |
|  |  |  | Region outside circle. |
|  |  |  | Region |
|  |  |  |  |
|  |  |  |  |

Example:

Sketch the region in the complex plane defied by all those complex numbers such that

and

For our problem:

tells us we don’t do any rotation

tells us we rotate radians



Don’t draw this line

Draw this line

Example: Sketch:

*Remember that is the same as*

<https://math.stackexchange.com/questions/782641/sketching-variations-of-argz>



For our problem:

tells us we don’t do any rotation

tells us we rotate radians, or an eighth of the plane

We also move the region one unit left

*s*

Example: Sketch:

*Remember that the modulus is smaller than the argument*

<https://www.doubtnut.com/question-answer/interpret-the-loci-arg-zpi-4-in-complex-plane-2109429>



For our problem:

tells us we don’t do any rotation

tells us we rotate radians, or an eighth of the plane

Example: Sketch:



**Lesson 4**

Exponential Form

and y

*euler’s formula*

*polar form in exponential form*

Example:

The complex number

It’s exponential form:

*also written as*

**Roots of complex numbers**

**Lesson 6**

Analytic Functions

<http://www.nhn.ou.edu/~milton/p5013/chap5.pdf>

Whenever exists, is said to be analytic (or regular, or holomorphic) at the point . The function is analytic throughout a region in the complex plane if exists for every point in that region. Any point at which does not exist or vanishes is called a singularity or singular point of the function f.

If is analytic everywhere in the complex plane, it is called entire.

Examples

• is analytic except at , so the function is singular at that point.

• The functions , a nonnegative integer, and are entire functions.

**Cauchy-Rieman equations**

The Cauchy-Riemann conditions are necessary and sufficient conditions for a

function to be analytic at a point.

[1] Show that a function is differentiable everywhere

[2] Verify that the **Cauchy-Rieman equations** are satisfied everywhere

Example: Show that the **Cauchy-Rieman equations** are never satisfied for , that is, if ; .

So is nowhere an analytic function of

Example: ASS 1 2021. Let g be defined by

, where

- Find all points where g is differentiable.

- Is analytic at any point of ? Give reasons for your answer.

**Cauchy-Rieman equations**

[1] Show that a function is differentiable everywhere

Suppose that . The complex-valued function is differentiable at any point z in the complex plane:

The real part and the imaginary part are

And their partial derivatives are

All the partial derivatives are polynomial and thus continuous. Therefore, the function is differentiable wherever the Cauchy-Rieman equations are satisfied

[2] Verify that the Cauchy-Rieman equationsare satisfied everywhere

Find all points where is differentiable

**Investigate where the Cauchy-Rieman equations are satisfied**

**i.e. where and**

If the Cauchy–Riemann equations are to hold at a point (x, y), it follows that and , or that .

We have that:

holds

Thus, holds:

or

Therefore, the Cauchy-Rieman equationsare satisfied at the lines

or .

OR

or √

To satisfy we need

√

This holds for all values of x and y. We require (2) and (3) to hold simultaneously. √

Therefore, the Cauchy-Rieman equations are satisfied only on the lines . √

For the function to be analytic at a point, it needs to be differentiable everywhere in some neighbourhood of that point. √

In this example, , , , exist everywhere and are continuous everywhere. √

It follows that is differentiable on the lines , √

Since the Cauchy-Rieman equations are not satisfied when , the function is not differentiable when √

[3] Verify if is analytic at any point of .

In order to be analytic, a function needs the partial derivatives of the real part and the imaginary part should:

- satisfy Cauchy-Rieman equations and , and

- be continuous

There is no neighborhood of any point throughout which is analytic as Cauchy-Rieman does not hold for an open set.

For the function to be analytic at a point, it needs to be differentiable everywhere in some neighbourhood of that point. √

In this example is only differentiable on the lines . √

If we take any neighbourhood of any point, then the neighbourhood will contain points not on the lines , i.e. points where is not differentiable. √

Therefore, is not analytic anywhere. √

**Lesson 7**

Harmonic Functions

A real-valued function H of two real variables x and y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

Or

**Laplace equation**

We can solve Laplace’s equation in any domain simply by taking the real part of any analytic function in that domain. Suppose that . The complex-valued function is differentiable at any point in the complex plane and the Cauchy-Rieman equations hold at any point ,

Assuming that is analytic in , we have that the first order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations throughout :

Differentiating with respect to , we have:

Differentiating with respect to , we have:

And also

and

That is, and are harmonic in

**Lesson 8**

Mapping by elementary functions

When , the image of the infinite strip is the upper half of the plane.

Let , where

Under the transformation ,

we have:

Under transformation where we have:

Under transformation where we have:

Which is a semi-circle in the upper half of the plane. The final sketch will not include points inside the circle.

So if and , then:

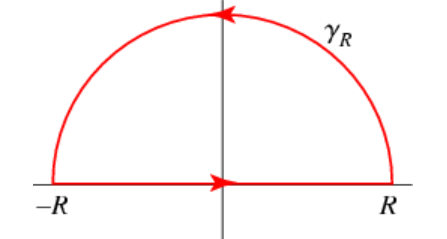
where

**Lesson 9**

Contour Integration

A contour in the complex plane is defined by a finite number of smooth curves

Contour integration is the process of calculating the values of a contour integral around a given contour in the complex plane. As a result of a truly amazing property of holomorphic functions, such integrals can be computed easily simply by summing the values of the complex residues inside the contour.



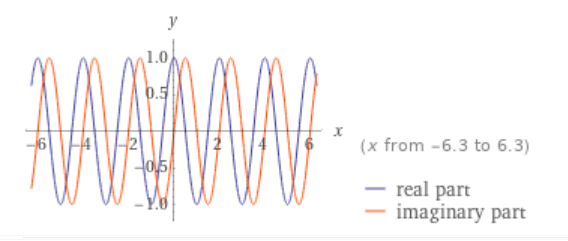
The contour is evaluated counter-clockwise

Example: integrate (semi-circle from lesson 8)

Wolfram

integrate+exp(i\*pi\*x)+dx

Result:



**Lesson 10**

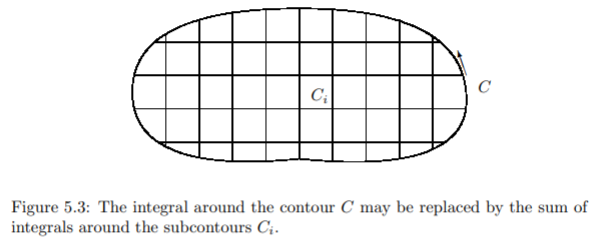
Cauchy’s integral formula

<http://www.nhn.ou.edu/~milton/p5013/chap5.pdf>

The Cauchy Integral Formula is probably the most important result in complex analysis. As a consequence of the Cauchy-Goursat Theorem, the Cauchy Integral Formula has some amazing consequences for holomorphic functions.

**Chauchy’s theorem**

states that if is analytic at all points on and inside a closed contour , then the integral of the function around that contour vanishes.



**Chauchy’s Integral Formula**

If (gamma) is simple, closed contour and is analytic on and inside :

*n is a non-negative integer*

The value of the function is irrelevant, except at

No need to integrate, only find some derivative of at

[1] Draw a diagram of the contour

[2] Draw in the singularities

*Any points at which does not exist or “vanishes”*

<https://www.youtube.com/watch?v=WJOf4PfoHow&ab_channel=MathsStatsUNSW>

Example:



*is the singularity, inside of (gamma)*

Using the formula

[1] multiply numerator by

[2] divide by

[3] multiply by the value of at



Example:

*is the singularity, outside of (gamma)*

*Irrelevant: we only want to look at analytic/inside gamma*

*is the singularity, inside of (gamma)*

*Just evaluate the function at*

Using the formula

[1] multiply numerator by

[2] divide by

[3] multiply by the value of at

Example:



*is the singularity, inside of (gamma)*

*Just evaluate the function at*

Using the formula

[1] multiply numerator by

[2] divide by

[3] multiply by the value of at

[NOV 2015]

**[1] Identify singularities (using numerator/bottom)**

**[3] Identify if singularities are inside/outside gamma**

*Use the integral limit b (bottom)*

*Substitute into b*

*Singularities where:*

*or*

**Inside gamma**

Only is inside

**Outside gamma**

N/A

**[2] Use the formula**

Wolfram

z^2+9

- multiply numerator by

- divide by

- multiply by the value of at

**Lesson 20**

Properties of z

**Modulus of z**

and

**Polar form**

,

*as would be undefined*

*cannot be negative, is the length of radius vector for*

; principal of

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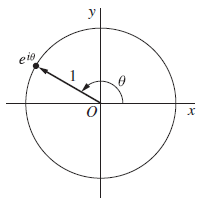
**Distance between two points**

*circle: centre*

**Exponential Form**

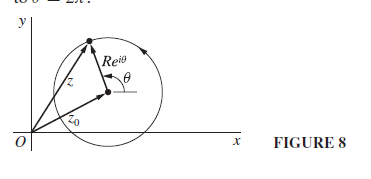
and y

*euler’s formula*

**

;

*polar form in exponential form*



;